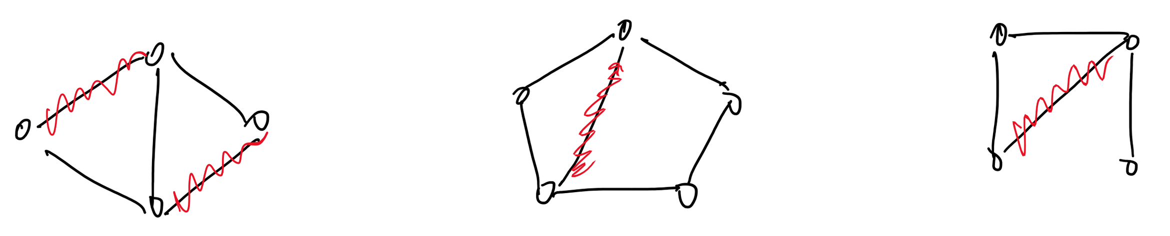
Greedy matchings

Monday, 2 September 2024 1:09 AM

Greedy algorithms:

Given a graph $G=(V,E)$, a matching $M \subseteq E$ is a set of edges s.t. no two edges in M are incident on the same vertex.



(a) (b) (c)

3 examples of matchings.

Now consider the following greedy algorithm to find matchings in graphs:

Algo Greedy Matching:

I/p: graph $G=(V,E)$

O/p: matching M

Let $M=\phi$ initially.

Order the edges $E=\{e_1, e_2, \dots, e_m\}$ arbitrarily

For $i=1 \dots m$

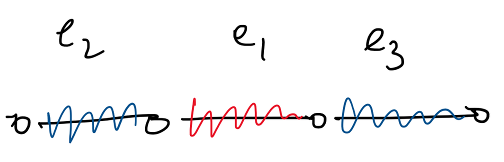
  if $M \cup \{e\}$ is a matching

  Add e to M.

How large is the matching obtained?

Do we always get the largest possible matching?

No! Consider the graph



The greedy algorithm could first pick the middle edge & stop, while the optimal (largest) matching consists of e_2 & e_3

Ok, so if M is matching obtained & M^* is largest matching, then possibly $|M| \leq \frac{1}{2}|M^*|$. Could it be even smaller?

Theorem: For any graph G, $|M| \geq \frac{1}{2}|M^*|$

That is, the greedy algorithm obtains a matching that is at least half the size of the optimal matching.

Proof:

Technique 1: Counting.

Consider an edge $e \in M^*$, $e \notin M$.

Since e wasn't picked in M, $\exists e' \in M \setminus M^*$ s.t. e, e' are incident on the same vtx.

Thus $\forall e \in M^* \setminus M$, define $f(e)$ = an arbitrary $e' \in M \setminus M^*$ s.t. e, e' are incident on the same vtx.

Note that there are at most 2 edges in M^* that are adjacent to an edge $e' \in M$. Hence $|M \setminus M^*| \geq \frac{1}{2}|M^* \setminus M|$

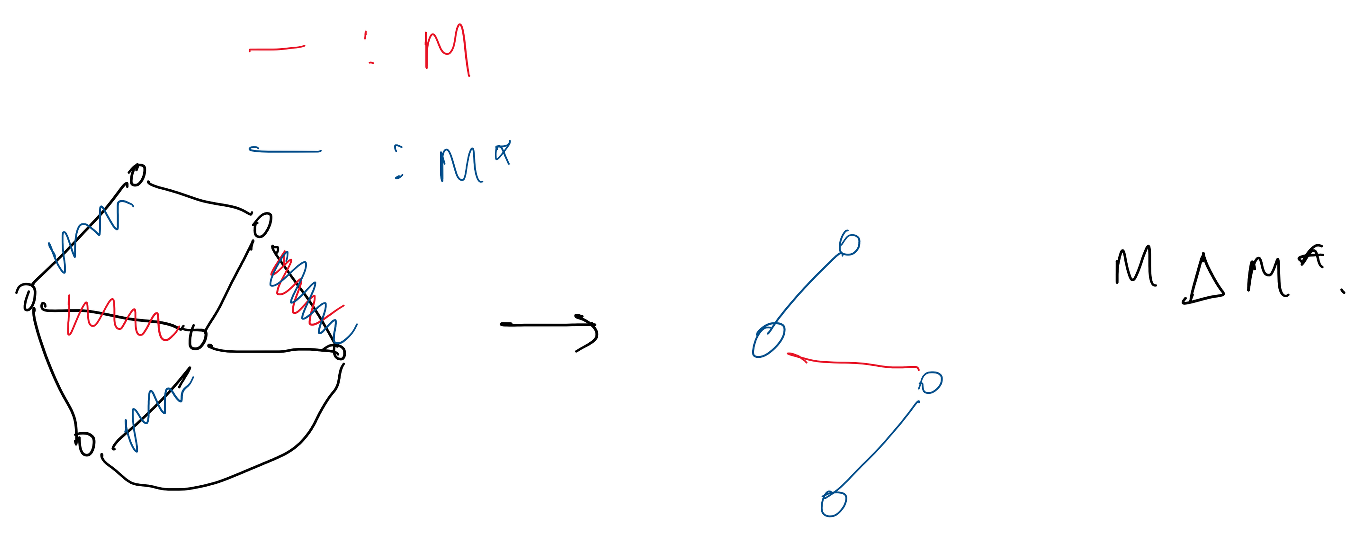
Hence, $|f^{-1}(e')| \leq 2 \quad \forall e' \in M \setminus M^*$

Thus, $|M^*| = |M \cap M^*| + |M^* \setminus M|$

$\leq |M \cap M^*| + 2|M \setminus M^*| \leq 2|M|$ ∎

Technique 2: Consider the symmetric difference $M^* \Delta M$. This consists of edges that are in exactly one of M^* & M

Example:



Note the following properties of $M \Delta M^*$:

① Every vtx. has degree ≤ 2, hence each component is a path or a cycle.

② M & M^* edges alternate: hence every cycle is an even cycle.

③ No component is a single edge. (prove why yourself).

Hence in every component C, $\frac{|M^* \cap C|}{|M \cap C|} \leq 2$

(in even paths & even cycles, $\frac{|M^* \cap C|}{|M \cap C|} = 1$, the bound of 2 is only achieved in a path of length three).

Then $\frac{|M^*|}{|M|} = \frac{|M^* \cap M| + \sum_C |M^* \cap C|}{|M^* \cap M| + \sum_C |M \cap C|} \leq 2$ ∎